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Structurally asymmetric d.c. SQUID's in the presence of coupling energy inhomogeneity of the junctions studied by means of the reduced two-junction interferometer model

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Abstract. The effects of structural asymmetries and of inhomogeneities in the junction parameters on the electrodynamic response of conventional d.c. SQUID's are studied both analytically and numerically. By a first-order perturbative expansion with respect to the parameter β and to the deviation parameters describing structural asymmetry and inhomogeneity, we write the reduced dynamical equation for the average superconducting phase difference of a conventional d.c. SQUID. As in homogeneous and symmetric SQUID's, the resulting dynamical equation is seen to be similar to that of a single junction with an unconventional current-phase relation characterized by a second harmonic contribution; in addition, a cosine term appears as a consequence of superconducting coupling inhomogeneity. By means of the reduced dynamical equation, the I-V characteristics in the presence of an external rf field and the critical current of the device are studied in the absence of noise.

PACS. 85.25.Dq Superconducting quantum interference devices (SQUIDs) – 74.50.+r Tunneling phenomena; point contacts, weak links, Josephson effects

1 Introduction

In order to describe the electrodynamic properties of d.c. SQUID's (Superconducting Quantum Interference Devices), the two-junction interferometer model can be adopted [1]. In this model, the coupled nonlinear ordinary differential equations governing the dynamics of the gauge-invariant superconducting phase differences across the Josephson junctions (JJ's) are often reduced to a single equation [2], setting to zero the parameter $\beta = \frac{LI_J}{\Phi_0}$, $2L$ being the inductance of the SQUID, I_J the average maximum Josephson current of the JJ and Φ_0 the elementary flux quantum. This simple assumption, valid in the small-β limit, greatly simplifies the analytic problem and captures most of the salient physical properties of the system [3]. On the other hand, the complete model can also be studied by means of numerical analysis [4].

Besides the great variety of applications that the superconducting devices allow [5], d.c. SQUID models are interesting dynamical systems *per se*. The periodic properties of the instantaneous voltage and the particular periodicity shown by the time-averaged voltage with respect to the externally applied flux, indeed, may justify all the theoretical effort made in studying these models. The behaviour of d.c. SQUID's in the presence of noise, of structural deformations or inhomogeneity in the junction parameters, is a well known subject in the literature [4]. New scenarios, however, have recently been opened by hybrid d.c. SQUID's, containing one conventional Josephson junction (0-junction) and one presenting an intrinsic phase difference of π (π-junction) [6,7].

The complete dynamical equations of this system show some very interesting peculiarities from the mathematical point of view. In this respect, the authors [8] have very recently shown that the complete set of dynamical equations written for a homogeneous system can be reduced to only one non-linear ordinary differential equation characterized by an unconventional current-phase relation (CPR) with a second harmonic contribution. Following this approach the amplitudes of the integer and half-integer steps, measured and calculated through numerical simulations by Vanneste et al. [9], have been found for small β-values.

In the present work we consider a structurally asymmetric SQUID with inhomogeneous junction parameters, as in a recent work by Müller et al. [10]. However, for this configuration, we only consider the very-low β limit in the absence of noise so that a reduced two-junction interferometer model can still be adopted. In the resulting non-linear ordinary differential equation for the average gauge-invariant superconducting phase difference across the Josephson junctions (JJ's) a second harmonic term appears because of the electromagnetic coupling between the two junctions. In addition, a cosine term appears as

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a consequence of superconducting coupling inhomogeneity of the two JJ's. On the basis of the reduced model we calculate, in closed analytic form, the amplitude of half-integer constant voltage steps in the current-voltage characteristics of the device. We notice that the above experimentally observable properties are first-order effects in the perturbation parameter β and in the inhomogeneity parameter ε. Moreover, the critical current of device is seen not to be affected, to first order in the perturbation parameters, by structural asymmetries and inhomogeneities in the junction parameters except for values of the applied external flux half-integer multiples of the elementary quantum flux Φ_0.

The paper is thus organized as follows. In the second section the reduced model dynamical equation for the average gauge-invariant superconducting phase difference is derived. In the third section the I-V characteristics of the device are studied and amplitudes of the constant-voltage steps are analytically derived. In the fourth section the effects of the structural asymmetries and inhomogeneities on the critical current are investigated. Conclusions are finally drawn in the last section.

2 The reduced two-junction interferometer model

Consider the two-junction interferometer circuital model schematically represented in Figure 1. If the Resistively Shunted Junction (RSJ) model [2] is adopted to describe the equations of the motion of the gauge-invariant superconducting phase differences φ_1 and φ_2 across the two junctions, the complete electrodynamic response of d.c. SQUID's can be given by solving this dynamical problem [2]. In the presence of structural asymmetry and of inhomogeneous junction parameters, however, some care must be taken to write down the time-evolution equation for φ_1 and φ_2. As we shall also notice, particular attention must also be paid in correctly defining observable electrodynamic quantities in terms of these functions. We thus start by assuming that the resistive parameters, the maximum Josephson currents and the branch inductances can be written, respectively, as follows:

$$R_1 = (1+\delta)\,R, \quad R_2 = (1-\delta)\,R \tag{1a}$$

$$I_{J1} = (1+\varepsilon)\,I_J, \quad I_{J2} = (1-\varepsilon)\,I_J \tag{1b}$$

$$L_1 = (1+\lambda)\,L, \quad L_2 = (1-\lambda)\,L. \tag{1c}$$

Here R, I_J, and L are the average values of the corresponding parameters, while the parameters δ, ε, and λ describe the deviation of the actual model parameters from these values. In the present analysis, we shall consider the deviation parameters small enough to allow a first-order perturbation analysis of the problem. In addition, we shall assume that the usual SQUID parameter $\beta = \frac{LI_J}{\Phi_0}$ is itself a perturbation parameter.

Following the schematic representation of the d.c. SQUID given in Figure 1, we briefly recall the main steps to obtain differential equations governing the dynamics

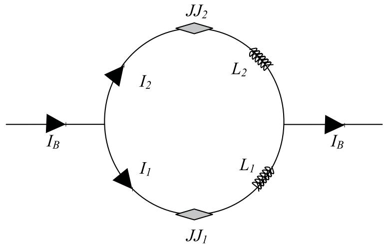


Fig. 1. Schematic representation of two-junction interferometer model.

of the gauge-invariant superconducting phase differences across the JJ's. We express the flux Φ as the sum of the applied flux Φ_{ex} plus the flux induced by the currents I_1 and I_2 in the two branches, so that

$$\Phi = \Phi_{ex} + L_1 I_1 - L_2 I_2. \tag{2}$$

By means of the above electrodynamic relation and by fluxoid quantization

$$\frac{2\pi}{\Phi_0}\Phi + \varphi_1 - \varphi_2 = 2\pi n, \tag{3}$$

n being an integer, the currents I_1 and I_2 can be related to the gauge-invariant superconducting phase differences as follows:

$$I_1 = \frac{1}{2}\left[(1-\lambda)\,I_B - \frac{\Phi_0}{2\pi L}(\varphi_1 - \varphi_2) - \frac{\Phi_{ex}}{L}\right] \tag{4a}$$

$$I_2 = \frac{1}{2}\left[(1+\lambda)\,I_B + \frac{\Phi_0}{2\pi L}(\varphi_1 - \varphi_2) + \frac{\Phi_{ex}}{L}\right] \tag{4b}$$

having taken $n = 0$.

By defining now the following normalized quantities

$$i_1 = \frac{I_1}{I_J}, \quad i_2 = \frac{I_2}{I_J}, \quad i_B = \frac{I_B}{I_J}, \quad \Psi_{ex} = \frac{\Phi_{ex}}{\Phi_0}, \quad \tau = \frac{2\pi R I_J}{\Phi_0} t \tag{5}$$

we can write:

$$\frac{1}{1+\delta}\frac{d\varphi_1}{d\tau} + (1+\varepsilon)\sin\varphi_1 + \frac{\varphi_1 - \varphi_2}{4\pi\beta} = \frac{1}{2}\left[(1-\lambda)\,i_B - \frac{\Psi_{ex}}{\beta}\right], \tag{6a}$$

$$\frac{1}{1-\delta}\frac{d\varphi_2}{d\tau} + (1-\varepsilon)\sin\varphi_2 - \frac{\varphi_1 - \varphi_2}{4\pi\beta} = \frac{1}{2}\left[(1+\lambda)\,i_B + \frac{\Psi_{ex}}{\beta}\right]. \tag{6b}$$

Introduce now the following new quantities: $\varphi_A = \frac{\varphi_1+\varphi_2}{2}$ and $\Psi = \frac{\varphi_2-\varphi_1}{2\pi}$, the first being simply the average value of the gauge-invariant superconducting phases, the second representing the number of fluxons trapped in to the

SQUID. The dynamical equations (6a-b) can be so rewritten in terms of the variables φ_A and Ψ:

$$\frac{d\varphi_A}{d\tau} + \cos(\pi\Psi)\sin\varphi_A - (\delta+\varepsilon)\sin(\pi\Psi)\cos\varphi_A - \frac{\delta}{2\beta}\Psi = \frac{i_B}{2} - \frac{\delta}{2\beta}\Psi_{ex}, \quad (7a)$$

$$\pi\frac{d\Psi}{d\tau} + \sin(\pi\Psi)\cos\varphi_A - (\delta+\varepsilon)\cos(\pi\Psi)\sin\varphi_A + \frac{\Psi}{2\beta} = (\lambda-\delta)\frac{i_B}{2} + \frac{\Psi_{ex}}{2\beta}. \quad (7b)$$

Notice that the above expressions are written disregarding terms which are of second order in the perturbation parameters. As in a previous work [8] we shall reduce the above set of equations to only one differential equation. We assume the existence of a first-order perturbed solution $\Psi(\tau)$ in the parameters β, δ, ε, and λ so that

$$\Psi(\tau) = \Psi_{ex} + \beta\Psi_\beta(\tau) + \delta\Psi_\delta(\tau) + \varepsilon\Psi_\varepsilon(\tau) + \lambda\Psi_\lambda(\tau). \quad (8)$$

By substituting equation (8) into equation (7b), we find, to first order in the perturbation parameters:

$$\Psi(\tau) = \Psi_{ex} - 2\beta\sin(\pi\Psi_{ex})\cos\varphi_A. \quad (9)$$

Substituting now the above expression in equation (7a) we have:

$$\frac{d\varphi_A}{d\tau} + \cos(\pi\Psi_{ex})\sin\varphi_A + \pi\beta\sin^2(\pi\Psi_{ex})\sin 2\varphi_A - \varepsilon\sin(\pi\Psi_{ex})\cos\varphi_A = \frac{i_B}{2}. \quad (10)$$

The above equation can be considered as a reduced two-junction interferometer model for a d.c. SQUID. In this model an effective non-sinusoidal CPR appears because of the non-vanishing contribution of correction terms linked to the parameter β and to the deviation parameter ε. In the absence of inhomogeneity in the Josephson energy coupling ($\varepsilon = 0$), we obtain the result already found in reference [8]. In this preliminary work, we have shown that, because of the additional second harmonic sine term in the effective CPR, half integer constant voltage steps appear in the I-V characteristics when the d.c. SQUID is subject to rf –frequency radiation.

From equation (10) we can finally notice that no effect is due to inhomogeneity in the resistive parameters and to structural asymmetry, according to the present first-order perturbation analysis.

3 Constant voltage steps in I-V characteristics

In the present section we derive the expression for the Shapiro steps in the I-V characteristics of a structurally asymmetric and inhomogeneous d.c. SQUID.

Consider the voltage V across the SQUID branches to be given by the following expression:

$$V(t) = V_0 + V_1\cos\omega_f t \quad (11)$$

where $\omega_f = 2\pi\nu$ is the angular frenquency of the rf signal. Introducing the normalized voltage $v(\tau) = \frac{V}{RI_J}$, we may write:

$$v(\tau) = \frac{d\varphi_A}{d\tau} + \pi\beta\lambda\frac{d}{d\tau}(i_1 - i_2). \quad (12)$$

Notice, however, that the second term in equation (12) is of order two with respect to the perturbation parameters. Considering equations (11) and (12), neglecting second order terms contributions, we can write:

$$\varphi_A(\tau) = \varphi_0 + \omega_0\tau + a\sin\omega\tau, \quad (13)$$

where $\omega_0 = \frac{V_0}{RI_J}$ and $a = \frac{V_1}{\nu\Phi_0}$ and $\omega = \frac{\Phi_0}{2\pi RI_J}\omega_f$. By substituting the expression for φ_A given in equation (13) into the stationary portion of equation (10) we get

$$I = 2\mathrm{Im}\left\{(x - i\varepsilon y)e^{i\varphi_0}e^{i\omega_0\tau}e^{ia\sin\omega\tau} + \pi\beta y^2 e^{i2\varphi_0}e^{i2\omega_0\tau}e^{i2a\sin\omega\tau}\right\}, \quad (14)$$

where the factor 2 is introduced to take account of the entire bias current, and where we have set $x = \cos(\pi\Psi_{ex})$ and $y = \sin(\pi\Psi_{ex})$. By now expanding the exponential of a sine as follows

$$e^{i\lambda\sin\omega t} = \sum_{k=-\infty}^{+\infty} J_k(\lambda)\, e^{ik\omega t}, \quad (15)$$

where J_k is the kth order Bessel function, we rewrite equation (14) as

$$I = 2\mathrm{Im}\left\{(x - i\varepsilon y)e^{i\varphi_0}\sum_{k=-\infty}^{+\infty} J_k(a)\, e^{i(\omega_0+k\omega)\tau} + \pi\beta y^2 e^{i2\varphi_0}\sum_{m=-\infty}^{+\infty} J_m(2a)\, e^{i(2\omega_0+m\omega)\tau}\right\}. \quad (16)$$

In order to find the cases in which non-vanishing time-averaged currents are present, we consider the following two propositions:

$$P \equiv \{\text{There exists an integer } k, \text{ such that } \omega_0 + k\omega = 0\};$$
$$Q \equiv \{\text{There exists an integer } m, \text{ such that } 2\omega_0 + m\omega = 0\}. \quad (17)$$

First of all notice that P implies Q, but Q does not necessarily imply P. Therefore, we only need to consider the following cases: a) Q is true; b) both P and Q are true. We shall now treat these cases one at a time.

Case a)

Since Q is true, we may write $m = -\frac{\omega_0}{\omega}$. We now distinguish the following two sub-cases.

i) $|m|$ even $\Rightarrow$ $m = -2p = -2\frac{\omega_0}{\omega}$, with p positive integer.

In this case both Q and P are true and we may set $k = -p$, so that the maximum value of the current step is obtained by maximizing, with respect to φ_0, the following expression:

$$I = 2\mathrm{Im}\left\{(-1)^p\,(x - i\varepsilon y)e^{i\varphi_0} J_p\,(a) + \pi\beta y^2 e^{i2\varphi_0} J_{2p}\,(2a)\right\}, \tag{18}$$

where we have used the following property of Bessel functions: $J_{-p}\,(x) = (-1)^p\,J_p\,(x)$. Detailed calculations are reported in Appendix, where it is shown that integer Shapiro steps are present in the I-V characteristics.

ii) $|m|$ odd $\Rightarrow$ $m = -\,(2q-1) = -2\frac{\omega_0}{\omega}$, with q positive integer.

In this case only Q is true, so that the maximum value of the current step is obtained by maximizing, with respect to φ_0, the following expression:

$$I = 2\mathrm{Im}\left\{-\pi\beta y^2 e^{i2\varphi_0} J_{2q-1}\,(2a)\right\}. \tag{19}$$

Here the maximum is readily found to be

$$\Delta I_{\frac{2q-1}{2}} = 2\pi\beta\sin^2\left(\pi\Psi_{ex}\right)\left|J_{2q-1}\,(2a)\right|, \tag{20}$$

and the steps having half-height $\Delta I_{\frac{2q-1}{2}}$ appear at $V_{0\left(\frac{2q-1}{2}\right)} = \frac{(2q-1)}{2}\Phi_0\nu$. Notice also that the expression for $\Delta I_{\frac{2q-1}{2}}$ given in equation (20) is at all identical to what obtained in the case $\varepsilon = 0$ [8].

Case b)

In this case both P and Q are true and we may set $m = 2k = -2p$, where p is a positive integer. The constant-voltage steps are thus obtained as in the previous case, with $|m|$ even. Detailed calculations for this case are performed in the Appendix.

In the following we summarize the results obtained in both cases:

i) Integer Shapiro steps appear at voltages $V_{0p} = p\Phi_0\nu$, where p is an integer, with amplitude

$$\Delta I_p = \begin{cases} 2\left|\cos\left(\pi\Psi_{ex}\right)J_p\,(a)\right| & \text{for } \Psi_{ex} \neq \frac{2k-1}{2} \\ 2\pi\beta\left|J_{2p}\,(2a)\right| + \sqrt{2}\varepsilon\left|J_p\,(a)\right| & \text{for } \Psi_{ex} = \frac{2k-1}{2}. \end{cases} \tag{21}$$

ii) Half-integer Shapiro steps appear at voltages $V_{0\left(\frac{2p-1}{2}\right)} = \frac{2p-1}{2}\Phi_0\nu$, where p is an integer, with amplitude

$$\Delta I_{\frac{2p-1}{2}} = 2\pi\beta\sin^2\left(\pi\Psi_{ex}\right)\left|J_{2p-1}\,(2a)\right|. \tag{22}$$

From the above it can be noticed what follows. When the present analysis is compared to the results obtained in reference [8] for a homogeneous d.c. SQUID with finite, but small, value of the parameter β, no correction to integer Shapiro steps to first order in ε is detectable, except for $\Psi_{ex} = \frac{2k-1}{2}$, where k is an integer. In this case experimental observation of an enhancement of the current step amplitudes given in equation (21) should be possible. On the other hand, according to the present perturbation analysis, half-integer Shapiro steps are not affected by structural asymmetry or inhomogeneity of the junction parameters.

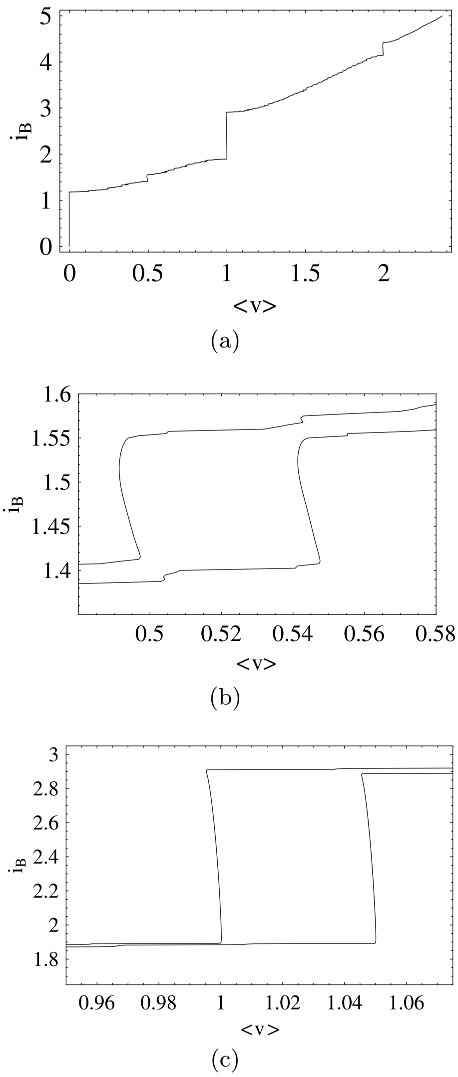


Fig. 2. (a) I-V characteristics for a d.c. SQUID with $\beta = 0.1$, $\varepsilon = 0$, $\Psi_{ex} = 0.25$, and $a = 0.9$. The voltage is normalized in such a way that steps appear at integer and half-integer values. (b) Comparison between the first half-integer steps, one obtained at $\varepsilon = 0$ (on the left), the other at $\varepsilon = 0.08$ (on the right). (c) Comparison between the first integer steps, one obtained at $\varepsilon = 0$ (on the left), the other at $\varepsilon = 0.08$ (on the right).

In Figure 2a we thus show the I-V characteristics for a d.c. SQUID with $\beta = 0.1$, $\varepsilon = 0$, $\Psi_{ex} = 0.25$, and $a = 0.9$. In this figure and in the following ones we normalize the voltage in such a way that steps appear at integer and half-integer values. For this choice of the field, we see that integer and half-integer steps are present. If we now let $\varepsilon = 0.08$, keeping the other parameters fixed, we should not be able, according to equation (21), to detect any variation in the steps. In Figure 2b we show the comparison

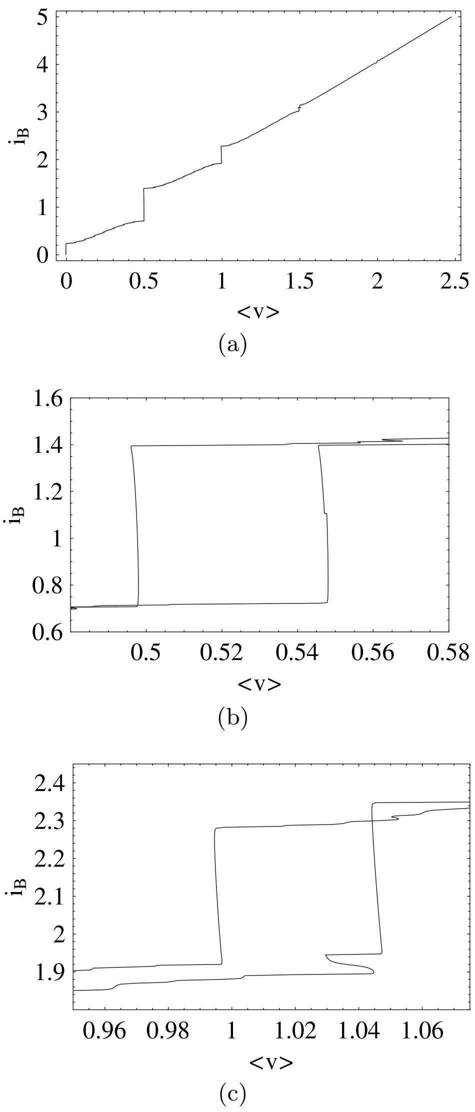


Fig. 3. (a) I-V characteristics for a d.c. SQUID with $\beta = 0.1$, $\varepsilon = 0$, $\Psi_{ex} = 0.5$, and $a = 0.9$. The voltage is normalized in such a way that steps appear at integer and half-integer values. (b) Comparison between the first half-integer steps, one obtained at $\varepsilon = 0$ (on the left), the other at $\varepsilon = 0.08$ (on the right). (c) Comparison between the first integer steps, one obtained at $\varepsilon = 0$ (on the left), the other at $\varepsilon = 0.08$ (on the right).

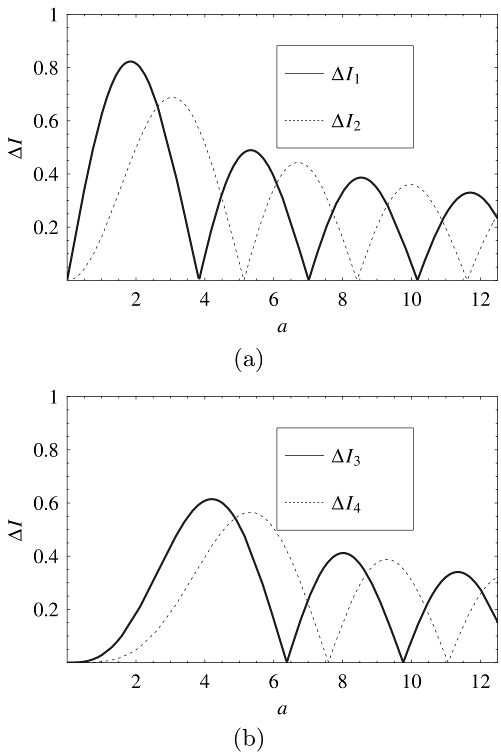


Fig. 4. Integer Shapiro steps as a function of variable a, proportional to the amplitude of the applied rf signal, for $\beta = 0.1$, $\varepsilon = 0.08$, $\Psi_{ex} = 0.25$. (a) Amplitude of first and second integer steps. (b) Amplitude of third and fourth integer steps.

between the first half-integer steps, one obtained at $\varepsilon = 0$, the other at $\varepsilon = 0.08$. In Figure 2c, instead, we show the comparison between the first integer steps, one obtained at $\varepsilon = 0$, the other at $\varepsilon = 0.08$. We find no variation in the step amplitudes.

In Figure 3a we show the I-V characteristics for a d.c. SQUID with $\beta = 0.1$, $\varepsilon = 0$, $\Psi_{ex} = 0.5$, and $a = 0.9$. For this choice of the field, half-integer steps attain the maximum value of the sine term depending on Ψ_{ex}. If we now let $\varepsilon = 0.08$, keeping the other parameters fixed, we should be able, this time, according to equation (21), to detect variation in integer steps. In Figure 3b we show the comparison between the first half-integer steps, one obtained at $\varepsilon = 0$, the other at $\varepsilon = 0.08$. We do not detect any variation in the amplitude of the step in this case. In Figure 3c we show the comparison between the first integer steps, one obtained at $\varepsilon = 0$, the other at $\varepsilon = 0.08$. In this last case we find an increase in the first integer step calculated for $\varepsilon = 0.08$, as predicted in equation (21), and detect an internal structure of the step itself.

In Figures 4a–b and in Figures 5a–b we report the amplitudes of the integer and half-integer Shapiro steps, respectively, as a function of a, which is proportional to the amplitude of the applied rf signal, for $\beta = 0.1$, $\varepsilon = 0.08$, $\Psi_{ex} = 0.25$. In particular, in Figure 4a the amplitudes of the first and the second integer steps are shown, while in Figure 4b the same is done for the third and the fourth integer steps. In Figure 5a the amplitudes of the first and the second half-integer steps are represented and, finally, in Figure 5b the amplitudes of the third and the fourth half-integer steps are shown. If we imagine to draw a vertical line passing through a fixed value of a in this figures, we see that there are various types of configurations for the integer and half-integer steps appearing in the I-V characteristics. For example, if we chose to run an experiment for $\beta = 0.1$, $\varepsilon = 0.08$, $\Psi_{ex} = 0.25$ and $a = 5.0$, we would expect the amplitude of the fourth integer step to be greater than the amplitudes of the first, second and third. On the other hand, the amplitude of the third and fourth half-integer steps are comparable for this choice of parameters, and greater than the amplitudes of the first and second half-integer steps. As for the choice of parameters in

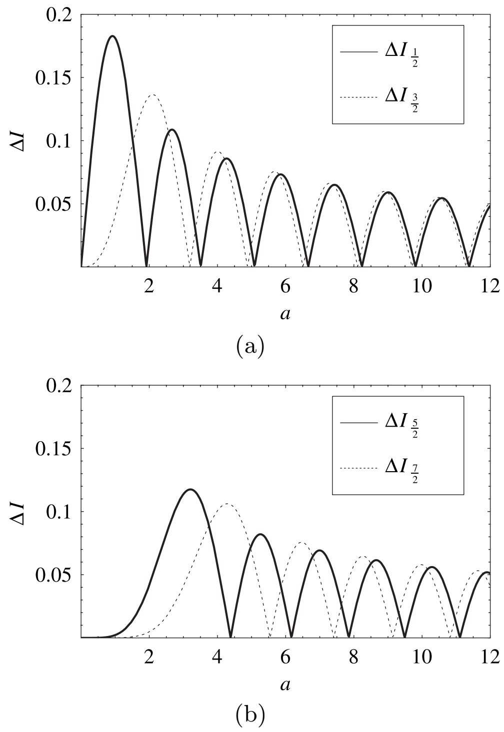


Fig. 5. Integer Shapiro steps as a function of variable a, proportional to the amplitude of the applied rf signal, for $\beta = 0.1$, $\varepsilon = 0.08$, $\Psi_{ex} = 0.25$. (a) Amplitude of first and second half-integer steps. (b) Amplitude of third and fourth half-integer steps.

Figure 2a, we notice that first integer step amplitude is about four times greater than the amplitude of the second integer step, as it can be argued from Figure 4a.

4 Critical current

In the present section we shall discuss on the possible corrections to the critical current of a d.c. SQUID due to the parameters β and ε, since, according to equation (10), only these parameters could give first order corrections to the electrodynamic quantities that are observable.

Let us first consider values of the normalized external flux equal to a half-integer value, i.e., $\Psi_{ex} = \frac{2k-1}{2}$, with k integer. We see that the equation of the motion of the d.c. SQUID (Eq. (10)) can be written in the following way:

$$\frac{d\varphi_A}{d\pi} + \pi\beta \sin 2\varphi_A - \epsilon \cos\varphi_A = \frac{i_B}{2}. \quad (23)$$

Notice that, for $\varepsilon = 0$, the normalize critical current of the device is $i_c = 2\pi\beta$. Similarly, for $\beta = 0$, we have $i_c = 2\varepsilon$. However, let us start considering a d.c. SQUID with $\beta \neq 0$, where ε is the perturbation parameter. In this way we can define a new perturbation parameter $\eta \equiv \frac{\varepsilon}{\pi\beta}$ so that the function to maximize with respect to φ_A is the following:

$$i_B = 2\pi\beta\left(\sin 2\varphi_A - \eta\cos\varphi_A\right). \quad (24)$$

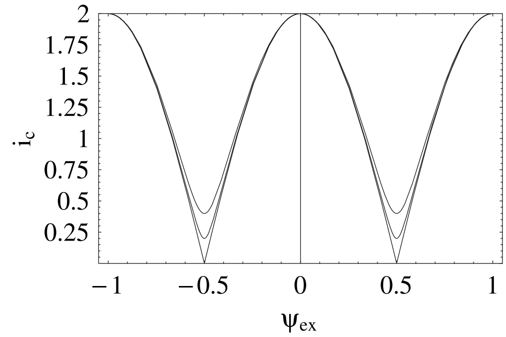


Fig. 6. i_c vs. Ψ_{ex} curves for $\varepsilon = 0,\ 0.05,\ 0.1$ (bottom to top) and for $\beta = 0$.

After some calculation we find:

$$i_c = 2\pi\beta + \sqrt{2}\varepsilon. \quad (25)$$

The above expression means that there is a first-order correction to the critical current for half-integer values of the normalized applied flux Ψ_{ex} either in β, either in ε. Notice, however, that we cannot reproduce the $\beta = 0$ result from equation (25), given the particular procedure followed [11].

Let us now consider the case $\Psi_{ex} \neq \frac{2k-1}{2}$, with k integer. It is well known that, for $\beta = 0$, the critical current of a d.c. SQUID in the presence of inhomogeneity in the maximum Josephson currents of the junctions, can be written as follows [2]:

$$I_c = \sqrt{I_{J1}^2 + I_{J2}^2 + 2I_{J1}I_{J2}\cos(2\pi\Psi_{ex})}. \quad (26)$$

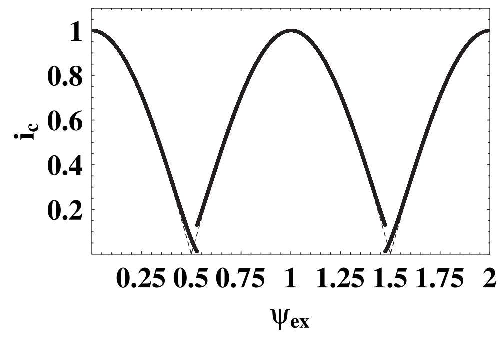
By substituting the expression for I_{J1} and I_{J2} in equation (1b), we find

$$\begin{aligned} i_c &= 2\sqrt{\cos^2\left(\pi\Psi_{ex}\right) + \varepsilon^2\sin^2\left(\pi\Psi_{ex}\right)} \\ &\approx 2\left|\cos\left(\pi\Psi_{ex}\right)\right|\left(1 + \frac{\varepsilon^2\sin^2\left(\pi\Psi_{ex}\right)}{2\cos^2\left(\pi\Psi_{ex}\right)}\right), \end{aligned} \quad (27)$$

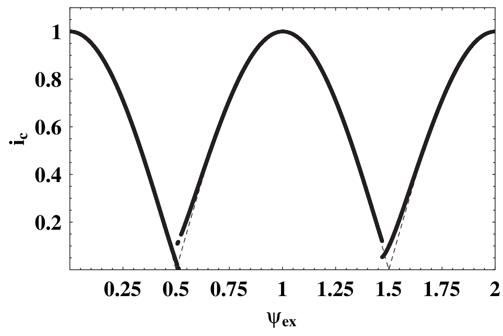
which proves absence of first-order correction in the parameter ε. The reason why there can be such a discrepancy in the behaviour of the critical current is clearly visible from Figure 6, where we report the plot of i_c vs. Ψ_{ex} for $\varepsilon = 0,\ 0.05,\ 0.1$, and for $\beta = 0$. In Figure 6 we notice that only in the vicinity of half-integer values of the normalized applied flux Ψ_{ex} the deviation from the unperturbed curve becomes relevant.

By now analyzing the effects of finite β-values on the critical current of the device, one can show that no correction to first order in β is present, except for $\Psi_{ex} = \frac{2k-1}{2}$, k integer, for which equation (25) gives the correction sought. For $\Psi_{ext} \neq \frac{2k-1}{2}$, on the other hand, a second order analysis is required. In Figure 7a and Figure 7b we show the curve obtained by numerically maximizing the expression

$$\begin{aligned} i_B = 2\left[\cos\left(\pi\Psi_{ex}\right)\sin\varphi_A + \pi\beta\sin^2\left(\pi\Psi_{ex}\right)\sin 2\varphi_A \right. \\ \left. -\varepsilon\sin\left(\pi\Psi_{ex}\right)\cos\varphi_A\right] \end{aligned} \quad (28)$$



(a)



(b)

Fig. 7. i_c vs. Ψ_{ex} curves for $\beta = 0.02$ and for: (a) $\varepsilon = 0$; (b) $\varepsilon = 0.05$.

as a function of Ψ_{ext}, for $\varepsilon = 0$ and for $\varepsilon = 0.05$, respectively, taking $\beta = 0.02$, superimposing these curves to the unperturbed one. We immediately notice two important features. First, we see once more that the unperturbed and perturbed curves collapse into a single curve at values of the normalized applied flux far enough form $\Psi_{ex} = \frac{2k-1}{2}$, k integer. The most significant effects of the perturbed model are thus given in the neighbourhood of $\Psi_{ex} = \frac{2k-1}{2}$, k integer. Second, we notice that these effects introduce a double periodicity in Ψ_{ex}, not allowing anymore a period $\Delta\Psi_{ex} = 1$ in the system, as for d.c. SQUID's with $\beta = 0$. This last aspect will be treated in more depth in a future work.

5 Conclusion

By a first-order perturbation approach we have written the reduced dynamical equation for the average superconducting phase difference of a structurally asymmetric conventional d.c. SQUID presenting inhomogeneities in the junction parameters. Following this approach we have calculated the amplitudes of integer and half-integer Shapiro steps appearing in the I-V characteristics of these devices in the presence of an external rf field.

Integer Shapiro steps are seen to appear at voltages $V_{0p} = p\Phi_0\nu$, where p is an integer, with amplitude

$$\Delta I_p = \begin{cases} 2\left|\cos\left(\pi\Psi_{ex}\right) J_p\left(a\right)\right| & \text{for } \Psi_{ex} \neq \frac{2k-1}{2} \\ 2\pi\beta\left|J_{2p}\left(2a\right)\right| + \sqrt{2}\varepsilon\left|J_p\left(a\right)\right| & \text{for } \Psi_{ex} = \frac{2k-1}{2}, \end{cases}$$

while the amplitudes of half-integer Shapiro steps, appearing at voltages $V_{0\left(\frac{2p-1}{2}\right)} = \frac{2p-1}{2}\Phi_0\nu$, p integer, are seen to take the following form:

$$\Delta I_{\frac{2p-1}{2}} = 2\pi\beta\sin^2\left(\pi\Psi_{ex}\right)\left|J_{2p-1}\left(2a\right)\right|.$$

In addition, by analysing, in the absence of noise, the critical current at small values of β and ε, the latter parameter describing coupling inhomogeneities of the junctions electrodes, we have noticed that the periodicity of this quantity with respect to the external normalized applied flux Ψ_{ex} doubles. This effect is a result of first-order corrections due to finiteness of β and ε at $\Psi_{ex} = \frac{2k-1}{2}$, k integer. In a future work we shall analyze the effects of noise on the above properties of the system.

Appendix

We start by considering the following expression, considering p as a positive integer throughout:

$$I = 2\text{Im}\left\{(-1)^p\left(x - i\varepsilon y\right)e^{i\varphi_0}J_p\left(a\right) + \pi\beta y^2 e^{i2\varphi_0}J_{2p}\left(2a\right)\right\}, \tag{A.1}$$

which is to be maximized with respect to φ_0. By setting to zero the derivative of I with respect to φ_0, we find

$$2\pi\beta y^2 J_{2p}\left(2a\right)z^4 + (-1)^p\left(x - i\varepsilon y\right)J_p\left(a\right)z^3 + (-1)^p\left(x + i\varepsilon y\right)J_p\left(a\right)z + 2\pi\beta y^2 J_{2p}\left(2a\right) = 0, \tag{A.2}$$

where $z = e^{i\varphi_0}$. Let us not solve exactly the above equation, since we are only interested to the zero and first-order term in the solution, so that we may set:

$$z = z_0 + \beta z_\beta + \varepsilon z_\varepsilon. \tag{A.3}$$

If we now substitute in equation (A.2) the above expression for z, by equating to zero the coefficients of the parameters β and ε, up to first order, we get

$$\begin{aligned} &z_0 = \pm i; \\ &(-1)^p\,xJ_p\left(a\right)z_\beta = 2\pi y^2 J_{2p}\left(2a\right); \\ &xJ_p\left(a\right)z_\varepsilon = -iz_0 yJ_p\left(a\right). \end{aligned} \tag{A.4}$$

Assume, in what follows, that $J_p\left(a\right) \neq 0$ and distinguish between two cases: $x = 0$ and $x \neq 0$.

For $x \neq 0$, solving equation (A.4) for z_β and z_ε, we have

$$z = \pm i + \beta\frac{2\pi y^2}{xJ_p\left(a\right)}J_{2p}\left(2a\right) \pm \varepsilon\frac{y}{x}. \tag{A.5}$$

In this way, substituting the above solution in equation (A.1), we find the maximum value of the pth current step to be:

$$\Delta I_p = 2\left|\cos\left(\pi\Psi_{ex}\right)J_p\left(a\right)\right|. \tag{A.6}$$

The above result signifies that, at arbitrary fields, i.e. for $\Psi_{ex} \neq \frac{2k+1}{2}$, k being an integer, ordinary integer current steps appear at the following voltage values

$$V_{0p} = p\Phi_0\nu \tag{A.7}$$

with no first-order correction to the expression for ΔI_p (Eq. (A.6)) already obtained in reference [8].

Next consider the case $x = 0$ and $y = 1$, which appears for $\Psi_{ex} = \frac{2k+1}{2}$, where k is an integer. We shall consider, in this case, from the very start, that β is finite and the only perturbation parameter is ε. For $\varepsilon = 0$, and assuming $J_{2p}(2a) \neq 0$, we need to consider the following equation:

$$z_1^4 + 1 = 0, \qquad (A.8)$$

whose solutions are $z_1 = \pm\sqrt{\pm i}$. For $\varepsilon \neq 0$, on the other hand, equation (A.2) can be written as follows:

$$2\pi\beta J_{2p}(2a)\, z^4 - i\varepsilon J_p(a)\, z^3 + i\varepsilon J_p(a)\, z + 2\pi\beta J_{2p}(2a) = 0. \qquad (A.9)$$

By introducing the new parameter $\alpha = \frac{J_p(a)}{2\pi J_{2p}(2a)}\frac{\varepsilon}{\beta}$, which we take to be much less than one, we consider solutions to the following simplified expression,

$$z^4 - i\alpha z^3 + i\alpha z + 1 = 0. \qquad (A.10)$$

To first order in α we find:

$$z = \pm\sqrt{\langle\pm\rangle\, i} + \frac{i\,\langle\mp\rangle\, 1}{4}\alpha, \qquad (A.11)$$

where the symbols $\langle\pm\rangle$ and $\langle\mp\rangle$ indicate that the order is fixed, so that when we choose the plus sign in the first, the minus sign must be chosen in the second, and vice versa. With this in mind we substitute equation (A.11) in the expression for the current in equation (A.1), considering $x = 0$ and $y = 1$, so that:

$$I = 2\pi\beta J_{2k}(2a)\,\mathrm{Im}\left\{z^2 + (-1)^{p+1}\, 2i\alpha z\right\} = \langle\pm\rangle\, 2\pi\beta J_{2k}(2a)\left[1 \pm \sqrt{2}\alpha\right]. \qquad (A.12)$$

The maximum of the above values, obtainable by four different choices of the signs, can finally be written as follows:

$$\Delta I_{2p} = 2\pi\beta\,|J_{2p}(2a)| + \sqrt{2}\varepsilon\,|J_p(a)|. \qquad (A.13)$$

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